الحل العددي للمعادلات التفاضلية العادية

Numerical solution of ordinary differential equations

In this chapter we will deal with the numerical methods for finding the solution of ordinary differential equation of first degree which is of the form:

$$\frac{dy}{dx} = F(x,y)$$

With the initial condition $y(x_0) = y_0$ which is called initial value problem.

We shall develop numerical methods for solution of the initial value problem in the interval $[a,b]$ on which the solution is derived in finite number of sub intervals by the points:

$$a = x_0 < x_1 < x_2 < x_3 \dots < x_n = b$$

The paints are called Mesh points. We assume that the points are spaced uniformly with the relation:

$$x_n = x_0 + nh$$

Taylor's series method

طريقة سلسلة تايلور

Let $y = f(x)$ be a solution of the equation

$$\frac{dy}{dx} = F(x, y)$$

with $y(x_0) = y_0$

Expanding $f(x)$ by Taylor's series about the point x_0 we get :

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots$$

$$y = y_0 + \frac{(x - x_0)}{1!} y'_0 + \frac{(x - x_0)^2}{2!} y''_0 + \cdots$$

putting $x = x_1 = x_0 + h$ we get

$$f(x_1) = y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0$$

we obtain:

$$y_{n+1} = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''$$

Example: solve $\frac{dy}{dx} = x + y$

if $y(1) = 0$

up to $x = 1.2$ with $h = 0.1$

solution:

we have $x_0 = 1$ $y_0 = 0$

$\frac{dy}{dx} = y' = x + y$ $y'_0 = 1 + 0 = 1$

$\frac{d^2y}{dx^2} = y'' = 1 + y'$ $y''_0 = 1 + 1 = 2$

$\frac{d^3y}{dx^3} = y''' = y''$ $y'''_0 = 2$

$\frac{d^4y}{dx^4} = y^{IV} = y'''$ $y^{IV} = 2$

$\frac{d^5y}{dx^5} = y^{V} = y^{IV}$ $y^{V} = 2$

$\vdots$

substituting the above values in:

$$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \frac{h^4}{4!} y^{IV}_0 + \frac{h^5}{5!} y^{V}_0$$

$$y_1 = 0 + 0.1 + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2) + \frac{(0.1)^5}{120}(2) \ldots$$

$$= 0.11033847$$

$y_1 = y(0.1) \approx 0.110$

Now

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$
$$y_1' = x_1 + y_1 = 1.1 + 0.110 = 1.21$$
$$y_1'' = 1 + y_1' = 1 + 1.21 = 2.21$$
$$y_1''' = y_1'' = 2.21$$
$$y_1^{IV} = 2.21$$
$$y_1^{V} = 2.21$$
$$\vdots$$

$$\therefore \quad y_2 = 0.110 + (0.1)(1.21) + \frac{(0.1)^2}{2}(2.21) + \frac{(0.1)^3}{6}(2.21) + \frac{(0.1)^4}{24}(2.21) + \frac{(0.1)^5}{120}(2.21)$$

$$= 0.24205$$

$$\vdots$$

$$\therefore \quad y(0.2) = 0.242$$

Example : Given $\frac{dy}{dx} = 1 + xy$ with the initial condition that $y = 1$, $x = 0$ compute $y(0.1)$ correct to four places of decimal by using Taylor's series method.

Solution :

$$\frac{dy}{dx} = 1 + xy \qquad y(0) = 1$$

$$y_0' = 1 + 0 * 1. = 1$$
$$y_0'' = y + x\frac{dy}{dx} = 1 + 0 \times 1 = 1$$
$$y_0''' = x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2$$

$$\frac{d^4y}{dx^4} = x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2}$$

$$y_0^{iv} = 3$$

$$\therefore y_1 = 1 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

$$y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(2) + \cdots$$

$$= 1.1053425$$

$$\therefore y(0.1) = 1.1053$$

Home Work :

;Apply the Taylor's Series method to find the value of $y(1.1)$ and $y(1.2)$ to three decimal places given that $\frac{dy}{dx} = xy^{1/3}$ and $y(1) = 1$ taking the first three terms of Taylor's series expansion.

ans. : $y(1.1) = 1.1066$
$y(1.2) = 1.228$